GEOMETRY QUALIFYING EXAM, SUMMER 2023

Instructions: Each problem is worth 20 points. You will be graded based on your best five problems but may submit solutions to more than five if you wish. You may refer to an earlier part in a problem to solve a later part even if you haven't solved the earlier part.

(1) Let $M = \{(x, y, z) \in \mathbb{R}^3 \,|\, z^2 - x^2 - y^2 = 1\}$ and consider the smooth vector field

$$X = xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + (x^2 + y^2)\frac{\partial}{\partial z}$$

defined on $\mathbb{R}^3$.

(a) Prove M is a regular two-dimensional submanifold of $\mathbb{R}^3$.

(b) Show that the restriction of X to M is everywhere tangent to M.

(c) Find a smooth vector field Y on $\mathbb{R}^3$ such that the restrictions of X and Y to M define a global framing of the tangent bundle of M.

(d) Evaluate the Lie bracket $[X, Y]$.

(2) Let $H = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$ denote the (open) upper-half plane and consider the orientation form $\omega \in \Omega^2(H)$ defined by $\omega = \frac{1}{y^2} dx \wedge dy$. Define a smooth map $F : H \to H$ by

$$F(x, y) = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}).$$

(a) Show that F is a diffeomorphism of H.

(b) Show that $F^*\omega = -\omega$.

(c) Let $D \subset H$ be the closed subset defined by

$$D = \{(x, y) \,|\, -1 \leq x \leq 1 \text{ and } \ x^2 + y^2 \geq 1.\}$$

Show $\int_D \omega = \pi$. You may use the antiderivative formula

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1}(x) + C.$$

(3) (a) State the inverse function theorem for smooth maps between smooth manifolds.

(b) Prove that if a 2x2 real matrix A is sufficiently close to the identity matrix then there exists a 2x2 real matrix B such that $B^2 = A$.

(4) (a) Define what an orientation is on a finite dimensional real vector space V. What are the possible number of orientations a finite dimensional real vector space can admit?

(b) Define what an orientation is on a smooth manifold M. What are the possible number of orientations a smooth and connected manifold can admit?

(c) Let M and N be smooth, connected, and oriented n-manifolds. Suppose that $U \subset \mathbb{R}^k$ is a connected open set and

$$F : U \times M \to N$$

is a smooth map such that for each $u \in U$, the smooth map

$$F_u : M \to N$$

defined by $F_u(m) = F(u, m)$ for each $m \in M$ is a diffeomorphism. Show that F_u is orientation-preserving for every $u \in U$ or F_u is orientation-reversing for every $u \in U$.

(5) (a) Define what it means for a smooth map $F : M \to N$ between smooth manifolds M and N to be a *submersion*.

(b) Prove that a submersion is an open map.

(c) Show there are no submersions from a compact manifold to a connected noncompact manifold.

(d) Does there exist a smooth submersion from $\mathbb{R}^2$ to $\mathbb{S}^2$? Justify your answer.

(6) (a) Define what it means for a smooth map $F : M \to N$ between smooth manifolds M and N to a be an *immersion*.

(b) Prove the map $F : \mathbb{S}^2 \to \mathbb{R}^4$ defined by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$ is an *immersion*.

(7) (a) State Stokes' theorem.

(b) Let M be a compact and oriented manifold with nonempty boundary ∂M. Prove there is no *retraction* from M to ∂M. That is, prove there is no smooth map $R : M \to \partial M$ such that $R(x) = x$ for each $x \in \partial M$.

(c) Give an example of an oriented manifold M with nonempty boundary ∂M and a retraction from M to ∂M.

(8) (a) Let X be a smooth vector field on a smooth manifold M. Define what it means for a smooth curve $c : I \to M$ to be an *integral curve* of X.

(b) Consider the smooth vector field on $\mathbb{R}^2$ defined by $X = e^{-x} \frac{\partial}{\partial x}$. Compute the maximal integral curve $c(t)$ of X with $c(0) = (x_0, y_0)$.